Static structure factor for graphene in a magnetic field

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A close study is made of the static structure factor for graphene in a magnetic field at integer filling factors ν, with focus on revealing possible signatures of "relativistic" quantum field theory in the low-energy physics of graphene. It is pointed out, in particular, that for graphene even the vacuum state has a nonzero density spectral weight, which, together with the structure factor for all ν, grows significantly with increasing wave vector; such unusual features of density correlations are a relativistic effect deriving from massless Dirac quasiparticles in graphene. Remarkably, it turns out that the zero-energy Landau levels of electrons or holes, characteristic to graphene, remain indistinguishable in density response from the vacuum state, although they are distinct in Hall conductance.

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I. INTRODUCTION

A great deal of attention has recently been directed to graphene, a monolayer of carbon atoms, both experimentally[1–3] and theoretically.[4–10] Graphene is marked with its novel charge carriers that behave like massless Dirac fermions with effective speed of light $v_F \approx 10^6$ m/s $\approx c/300$. It, thus, provides a special opportunity to study "relativistic" quantum dynamics in condensed-matter systems. Experiments have revealed a number of exotic transport properties of graphene, such as the half-integer quantum Hall (QH) effect and minimal conductivity.

The dynamics of Dirac fermions becomes particularly interesting in a magnetic field, and leads to peculiar quantum phenomena, such as fermion number fractionalization[11] and spectral asymmetry,[12–14] intimately tied to the chiral anomaly in 1+1 dimensions. Actually, the half-integer QH effect and the presence of the zero-energy Landau levels observed[1,2] in graphene are a manifestation of fermion number fractionalization.

It would be important to explore further possible signatures of relativistic quantum field theory in the low-energy physics of graphene. An interesting proposal[15] along this direction is to simulate the Klein paradox[16] (or tunneling) in graphene. Calculations[17] of the dielectric function also reveal that the electromagnetic response of graphene is substantially different from that of conventional two-dimensional systems. The difference becomes even prominent under a strong magnetic field.[18] In particular, for graphene, the vacuum state is a dielectric medium and carries an appreciable amount of electric and magnetic susceptibilities over all range of wavelengths; this reflects the presence of the "Dirac sea." Curiously, the zero-energy Landau levels, though distinct in Hall conductance, hardly contribute to the susceptibilities.

The purpose of this paper is to study further aspects of the response of graphene in a magnetic field at integer filling factors ν, with focus on the static structure factor $s(\mathbf{p}) \sim \langle \rho_{-\mathbf{p}} \rho_{\mathbf{p}} \rangle$, which is directly related to the cross section in inelastic light scattering by graphene. In particular, we point out that graphene has unusual characteristics of electronic correlations at short distances, which derive from the relativistic nature of massless quasiparticles. It is also shown that the zero-energy Landau levels remain indistinguishable in spectral weight $\langle \rho_{-\mathbf{p}} \rho_{\mathbf{p}} \rangle$ from the vacuum state.

In Sec. II, we briefly review the low-energy effective theory of graphene in a magnetic field and derive, as a preliminary, the structure factor for conventional QH systems. In Sec. III, we study the structure factor for graphene. Section IV is devoted to a summary and discussion.

II. LOW-ENERGY EFFECTIVE THEORY

Graphene has a honeycomb lattice consisting of two triangle sublattices of carbon atoms with one electron per site. It is a gapless semiconductor and its low-energy electronic transport is described by an effective Hamiltonian of the form[19]

$$H = \int d^2\mathbf{x} [\psi^\dagger \mathcal{H}_+ \psi + \chi^\dagger \mathcal{H}_- \chi],$$

$$\mathcal{H}_\pm = v_F(\sigma_1 \Pi_1 + \sigma_2 \Pi_2 \pm m\sigma_3) - eA_0, \tag{2.1}$$

where $\Pi_i = -i\partial_i + eA_i$ $[i=(1,2)$ or $(x,y)]$ includes coupling to external electromagnetic potentials $A_\mu = (A_i, A_0)$; $v_F \sim 10^6$ m/s is the Fermi velocity. The two-component spinors $\psi = (\psi_1, \psi_2)^t$ and $\chi = (\chi_1, \chi_2)^t$ stand for the electron fields near the two inequivalent Fermi points (K and K') where the spectrum becomes linear; (ψ_1, χ_2) reside on the same sublattice and (ψ_2, χ_1) on another.

For generality, we have introduced a tiny "mass" gap $m > 0$, which spoils the valley SU(2) symmetry of H. Actually, the observed $\nu = \pm 1$ Hall plateaus[3] suggest such a tiny mass gap.[8–10] We keep $m \neq 0$ to clarify the particle-hole character of the lowest Landau levels, but practically set $m \to 0$.

We suppress the electron spin, which is treated as a global SU(2) symmetry of H, by doubling the fields ψ and χ. The Zeeman splitting, though ignored for simplicity, is readily incorporated.

The Coulomb interaction is written as

$$H^{\rm Coul} = \frac{1}{2}\sum_{\mathbf{p}} v_{\mathbf{p}}\rho_{-\mathbf{p}}\rho_{\mathbf{p}}, \tag{2.2}$$

where $\rho_{\mathbf{p}}$ is the Fourier transform of the electron number density $\rho=\psi^\dagger\psi+\chi^\dagger\chi$; $v_{\mathbf{p}}=2\pi\alpha/(\epsilon_b|\mathbf{p}|)$ is the Coulomb potential with the fine-structure constant $\alpha=e^2/4\pi\epsilon_0\approx 1/137$ and the substrate dielectric constant ϵ_b. We shall discuss the effect of $H^{\rm Coul}$ later.

Let us place graphene in a strong magnetic field and study how the electrons in graphene respond to a weak potential $A_0(x)$. We set $A_i\to B(-y,0)$ to supply a uniform magnetic field $B_z=B>0$ normal to the sample plane.

When $A_0=0$, the eigenmodes of H are Landau levels of ψ and χ of energy

$$\epsilon_n = s_n\omega_c\sqrt{|n|+m^2\ell^2/2}, \tag{2.3}$$

labeled by integers $n=0,\pm1,\pm2,\ldots$, and p_x (or $y_0\equiv\ell^2p_x$, with the magnetic length $\ell\equiv1/\sqrt{eB}$); $\omega_c=\sqrt{2}v_F/\ell$ is the basic cyclotron frequency. Here, $s_n\equiv{\rm sgn}\{n\}=\pm1$ specifies the sign of the energy ϵ_n.

For $n\neq0$, ψ and χ have the same spectrum symmetric about $\epsilon=0$. The $n=0$ level of ψ has negative energy $\epsilon_{0_-}=-v_Fm$, while that of χ has positive energy $\epsilon_{0_+}=v_Fm$; these $n=0_\mp$ levels represent holes and electrons via quantization. With the electron spin taken into account, each Landau level is thus fourfold degenerate, except for the doubly degenerate $n=0_\pm$ levels. The $n=0_\pm$ eigenmodes have components only on each separate sublattice.

To make this Landau-level structure explicit, it is useful to pass to the $|n,y_0\rangle$ basis, with the expansion[20] $\psi(\mathbf{x},t)=\Sigma_{n,y_0}\langle\mathbf{x}|n,y_0\rangle\psi_n(y_0,t)$. (From now on, we shall only display the ψ sector since the χ sector is obtained by reversing the sign of m.) The Hamiltonian H thereby is rewritten as

$$H=\int dy_0\sum_{n=-\infty}^{\infty}\psi_n^\dagger\epsilon_n\psi_n, \tag{2.4}$$

and the charge density $\rho_{-\mathbf{p}}(t)=\int d^2\mathbf{x}e^{i\mathbf{p}\cdot\mathbf{x}}\psi^\dagger\psi$ as[18]

$$\rho_{-\mathbf{p}}=e^{-\ell^2\mathbf{p}^2/4}\sum_{k,n=-\infty}^{\infty}g^{\psi}_{kn}(\mathbf{p})\int dy_0\psi_k^\dagger e^{i\mathbf{p}\cdot\mathbf{r}}\psi_n, \tag{2.5}$$

with the coefficient matrix

$$g^{\psi}_{nn'}(\mathbf{p})=\tfrac{1}{2}[c_n^+c_{n'}^+f_{|n|-1,|n'|-1}(\mathbf{p})+s_ns_{n'}c_n^-c_{n'}^-f_{|n|,|n'|}(\mathbf{p})] \tag{2.6}$$

and $c_n^\pm=\sqrt{1\pm v_Fm/\epsilon_n}$; $\mathbf{r}=(r_1,r_2)=(i\ell^2\partial/\partial y_0,y_0)$ stands for the center coordinate with uncertainty $[r_1,r_2]=i\ell^2$. Here, the coefficient functions

$$f_{kn}(\mathbf{p})=(k|e^{-i(lp/\sqrt{2})a^\dagger}e^{-i(lp^\dagger/\sqrt{2})a}|n) \tag{2.7}$$

are defined in terms of the harmonic oscillator eigenstates $\{|n)\}$ with $a^\dagger a|n)=n|n)$ and $[a,a^\dagger]=1$; $p=p_y+ip_x$ and $p^\dagger=p_y-ip_x$. More explicitly,[21]

$$f_{kn}(\mathbf{p})=\sqrt{\frac{n!}{k!}}\left(\frac{i\ell p}{\sqrt{2}}\right)^{k-n}L_n^{(k-n)}\left(\frac{1}{2}\ell^2\mathbf{p}^2\right) \tag{2.8}$$

for $k\geqslant n$, and $f_{nk}(\mathbf{p})=[f_{kn}(-\mathbf{p})]^\dagger$. Actually, $f_{kn}(\mathbf{p})$ are the coefficient functions that characterize the charge density for the "nonrelativistic" Hall electrons

$$\rho_{-\mathbf{p}}=e^{-\ell^2\mathbf{p}^2/4}\sum_{k,n=0}^{\infty}f_{kn}(\mathbf{p})\int dy_0\psi_k^\dagger e^{i\mathbf{p}\cdot\mathbf{r}}\psi_n \tag{2.9}$$

expressed in terms of their eigenmodes $\{\psi_n(y_0,t)\}$ with energy $\epsilon_n=(eB/m^*)(n+1/2)$.

Let us first consider, as an exercise, the static structure factor or the spectral weight $\langle\rho_{-\mathbf{p}}\rho_{\mathbf{p}}\rangle\equiv\langle G|\rho_{-\mathbf{p}}\rho_{\mathbf{p}}|G\rangle$ for a state $|G\rangle$ of free nonrelativistic Hall electrons with integer filling factor ν (per spin). One may use Eq. (2.9) and note, in taking the matrix element $\langle G|\rho_{-\mathbf{p}}\rho_{\mathbf{p}}|G\rangle$, that $\psi_{k'}(y_0',t)\psi_k^\dagger(y_0,t)\to\delta_{k'k}\delta(y_0'-y_0)$ for unoccupied levels (k',k) so that the result is proportional to the number of electrons per occupied level $\int dy_0\psi_n^\dagger\psi_n\to L_xL_y/2\pi\ell^2$. [Note here that $\delta(y_0=0)=L_x/(2\pi\ell^2)$, with $L_x=\int dx$.] This yields the spectral weight

$$\langle\rho_{-\mathbf{p}}\rho_{\mathbf{p}}\rangle=\frac{\Omega}{2\pi\ell^2}e^{-\ell^2\mathbf{p}^2/2}\sum_{k=\nu}^{\infty}\sum_{n=0}^{\nu-1}|f_{kn}(\mathbf{p})|^2 \tag{2.10}$$

for $\mathbf{p}\neq0$, where $\Omega=L_xL_y$ denotes the total area.

Let us recall that the static structure factor[22]

$$s(\mathbf{p})=(\langle\rho_{-\mathbf{p}}\rho_{\mathbf{p}}\rangle-N_e^2\delta_{\mathbf{p},0})/N_e \tag{2.11}$$

is defined by $\langle\rho_{-\mathbf{p}}\rho_{\mathbf{p}}\rangle$ with its $\mathbf{p}=0$ component isolated, where $N_e=\nu\Omega/2\pi\ell^2$ stands for the total electron number. Accordingly, in general, $s(\mathbf{p}\to0)=0$ owing to charge conservation. Noting the formula

$$\sum_{k=0}^{\infty}|f_{kn}(\mathbf{p})|^2=e^{\ell^2\mathbf{p}^2/2} \tag{2.12}$$

allows one to cast Eq. (2.10) into the structure factor at filling factor ν,

$$s(\mathbf{p})=1-e^{-\ell^2\mathbf{p}^2/2}\frac{1}{\nu}\sum_{n=0}^{\nu-1}\sum_{k=0}^{\nu-1}|f_{kn}(\mathbf{p})|^2, \tag{2.13}$$

which agrees with a known result.[23] In particular,

$$s(\mathbf{p})|_{\nu=1}=1-e^{-\ell^2\mathbf{p}^2/2}. \tag{2.14}$$

Note first that $s(\mathbf{p}\to0)\to0$ since $f_{kn}(0)=\delta_{kn}$. Note also that $s(\mathbf{p})\to1$ as $\mathbf{p}\to\infty$ for all ν. To see what this means, let us recall the following: For a collection of classical particles, $s(\mathbf{p})$ is written as[22]

$$s(\mathbf{p})-1=\langle\sum_{\Delta\mathbf{r}}e^{i\mathbf{p}\cdot\Delta\mathbf{r}}\rangle \tag{2.15}$$

for $\mathbf{p}\neq0$, i.e., as an average over the relative positions $\Delta\mathbf{r}$ of particles surrounding a given particle. As a result, $s(\mathbf{p})\to1$ for $\mathbf{p}\to\infty$ if $\Delta\mathbf{r}\neq0$, e.g., for particles formulated on a lattice. Such behavior of $s(\mathbf{p})$, thus, implies the absence of particle correlations at short distances.

III. STATIC RESPONSE OF GRAPHENE

In this section, we study the case of graphene. In the present treatment, the ψ and χ sectors are independent, and the spectral weight $\langle \rho\rho \rangle$ with $\rho = \rho^{\psi} + \rho^{\chi}$ is given by the sum $\langle \rho^{\psi}\rho^{\psi} \rangle + \langle \rho^{\chi}\rho^{\chi} \rangle$ for $\mathbf{p} \neq 0$. For the ψ sector, one may simply replace, in Eq. (2.10), $f_{kn}(\mathbf{p})$ by $g^{\psi}_{kn}(\mathbf{p})$ of Eq. (2.6) and note that the level indices (k,n) now run over all integers, $0_{-}, \pm 1, \pm 2, \ldots$. Analogously, $g^{\chi}_{kn}(\mathbf{p})$ for the χ sector is obtained from $g^{\psi}_{kn}(\mathbf{p})$ by setting $m \to -m$ (i.e., $c^{+}_{n} \leftrightarrow c^{-}_{n}$) or, equivalently,

$$g^{\chi}_{kn}(\mathbf{p}) = g^{\psi}_{-k,-n}(\mathbf{p}). \tag{3.1}$$

Let us denote $\Gamma^{\psi}_{kn}(z) = |g^{\psi}_{kn}(\mathbf{p})|^2$ and $\Gamma^{\chi}_{kn}(z) = |g^{\chi}_{kn}(\mathbf{p})|^2$ for short. They actually are functions of $z = \ell^2 \mathbf{p}^2/2$, are thus symmetric in (k,n), and have the property

$$\Gamma^{\psi}_{kn} = \Gamma^{\chi}_{-k,-n}, \quad \Gamma^{i}_{kn} = \Gamma^{i}_{nk}, \tag{3.2}$$

with $i = \psi$ and χ. The spectral weight is now written as

$$\langle \rho_{-\mathbf{p}}\rho_{\mathbf{p}} \rangle / \Omega = (\lambda_s / 2\pi\ell^2) e^{-(1/2)\ell^2\mathbf{p}^2} (I^{\psi} + I^{\chi}), \tag{3.3}$$

where $I^{i} = \Sigma'_{k}\Sigma_{n}\Gamma^{i}_{kn}(z)$; $\lambda_s = 2$ stands for the spin degeneracy. One can write I^{ψ} as

$$I^{\psi}_{j}(z) = \sum_{k>j}\sum_{n\le j} \Gamma^{\psi}_{kn}(z), \tag{3.4}$$

when the ψ Landau levels are occupied up to the jth level ($j = 0_{-}, \pm 1, \ldots$); analogously for I^{χ}_{j}.

Note that Eq. (3.1) relates I^{ψ} and I^{χ} so that

$$I^{\psi}_{j} = I^{\chi}_{-(j+1)}. \tag{3.5}$$

The equalities $I^{\psi}_{0_-} = I^{\chi}_{-1}$ and $I^{\psi}_{j} + I^{\chi}_{j} = I^{\psi}_{-(j+1)} + I^{\chi}_{-(j+1)}$ then imply the following: (i) The ψ and χ sectors contribute equally to the vacuum spectral weight, $\langle \rho_{-\mathbf{p}}\rho_{\mathbf{p}} \rangle|_{\nu=0} \propto I^{\psi}_{0_-} + I^{\chi}_{-1} = 2I^{\psi}_{0_-}$. (ii) The spectral weight $\langle \rho_{-\mathbf{p}}\rho_{\mathbf{p}} \rangle$ is the same for the charge-conjugate states with $\nu = \pm 2(2j+1)$. In view of this, we shall focus on the case $\nu \geqslant 0$ from now on.

In the limit $m \to 0$ of practical interest, distinctions between Γ^{ψ} and Γ^{χ} disappear: $\Gamma^{\psi}_{kn} = \Gamma^{\chi}_{kn} = \Gamma^{\chi}_{-k,-n}$ for $kn \neq 0$ and $\Gamma^{\psi}_{k,0_-} = \Gamma^{\chi}_{\pm k,0_+}$ for $k \neq 0$, as seen from Eq. (2.6). This yields

$$I^{\psi}_{j} = I^{\chi}_{j} = I^{\psi}_{-(j+1)} = I^{\chi}_{-(j+1)} \quad (m \to 0). \tag{3.6}$$

In particular, $I^{\psi}_{0_-} = I^{\chi}_{0_+} = I^{\chi}_{-1} = I^{\psi}_{-1}$ shows that the graphene vacuum and the $\nu = \pm 2$ states, all of zero energy, have the same spectral weight $\langle \rho_{-\mathbf{p}}\rho_{\mathbf{p}} \rangle|_{\nu=2} \propto I^{\psi}_{0_-} + I^{\chi}_{0_+} = 2I^{\psi}_{0_-}$. Actually, we earlier noted such degeneracy in electric susceptibility for the $\nu = 0$ and $\nu = \pm 2$ states,[18] and these states now turn out to be indistinguishable in both the real and imaginary parts of density response; we discuss this point in more detail later.

To proceed further, let us note some basic properties of the transition rates $\Gamma^{i}_{kn}(z)$ with $z = \ell^2\mathbf{p}^2/2$. It is clear from Eq. (3.2) that the transition rates among the positive-energy states and those among the negative-energy states are essentially the same, whereas they are different from the rates across the Dirac sea ($\propto \Gamma_{kn}$ with $kn < 0$). This feature is seen from Fig. 1(a), which illustrates typical profiles of $e^{-z}\Gamma_{kn}(z)$ for $m \to 0$. In general, $e^{-z}\Gamma^{i}_{kn}(z)$ is significantly peaked at

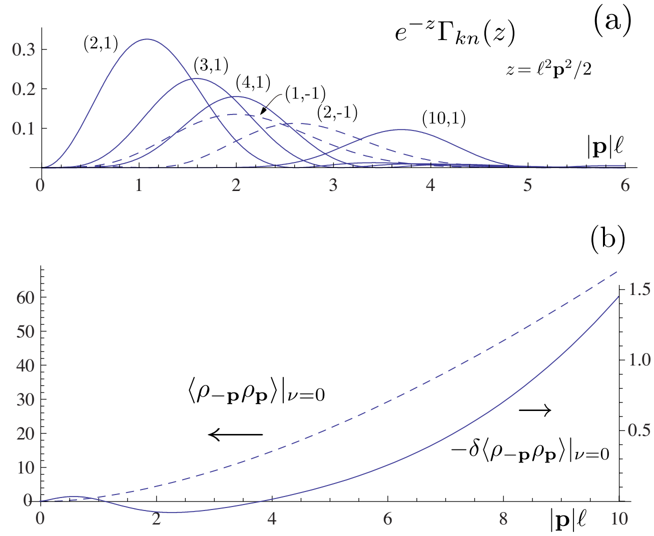

FIG. 1. (Color online) (a) Profile of the transition rate $e^{-z}\Gamma^{\psi}_{kn}(z)$ for $m \to 0$, labeled with (k,n). (b) Vacuum spectral weight $\langle \rho_{-\mathbf{p}}\rho_{\mathbf{p}} \rangle|_{\nu=0}$ for $N = 800$ (dashed line) and the difference $-\delta\langle \rho_{-\mathbf{p}}\rho_{\mathbf{p}} \rangle|_{\nu=0} = \langle \rho\rho \rangle^{B=0} - \langle \rho\rho \rangle^{B\neq 0}$ (real line), both in units of $\lambda_s\Omega/2\pi\ell^2$.

some value of $\ell|\mathbf{p}|$ and the peak position rises only gradually as the level gap $\sim |k-n|$ increases. The structure factor $s(\mathbf{p})$ for small $|\mathbf{p}|$ is, thus, governed by virtual transitions to neighboring levels, while its property at larger $|\mathbf{p}|$ is determined by transitions across larger gaps.

It is readily expected from this property that the sum over an infinite number of negative-energy levels, would make the weight $\langle \rho_{-\mathbf{p}}\rho_{\mathbf{p}} \rangle$ cutoff (N) dependent for small $\mathbf{p}^2$, actually to $O(\ell^2\mathbf{p}^2)$. In particular, the vacuum weight $\langle \rho_{-\mathbf{p}}\rho_{\mathbf{p}} \rangle|_{\nu=0} \propto 2e^{-z}\Sigma^{N}_{k=1}\Sigma^{N}_{n=0}\Gamma^{\psi}_{k,-n}(z)$ vanishes at $z = 0$ and grows rapidly with z for fixed N, whereas it diverges as $N \to \infty$ for $z \neq 0$ [see Fig. 1(b)]. Indeed, evaluating the $O(z)$ terms in $\Gamma_{k,-n}(z)$ shows that the divergence is logarithmic in N,

$$\sum_{k=1}^{N}\sum_{n=0}^{N} \Gamma^{\psi}_{k,-n} \overset{O(z)}{\approx} \frac{1}{2} z \sum_{k=0}^{N-1} (\sqrt{k+1} - \sqrt{k})^2 \approx \frac{1}{8} z \ln(c_1 N), \tag{3.7}$$

with $c_1 \approx 54.088$ and for $m \to 0$. This ultraviolet divergence is physical. The infinite depth of the Dirac sea, of course, is an artifact of the continuum model (2.1), and the cutoff scale $\omega_c\sqrt{N}$ is set by the energy scale above which the model loses its validity.

Let us recall here that for conventional QH systems, the charge operator trivially annihilates the vacuum, $\rho|\nu=0\rangle = 0$, and the vacuum spectral weight vanishes. Accordingly, the nonzero vacuum weight $\langle \rho_{-\mathbf{p}}\rho_{\mathbf{p}} \rangle|_{\nu=0}$ itself is a relativistic signature of graphene,

$$\rho(x)|\nu = 0\rangle_{\text{graphene}} \neq 0, \tag{3.8}$$

which is a consequence of particle-hole pair creation or of the presence of the Dirac sea.

It is perfectly legitimate to consider the vacuum weight with such a physical cutoff N (apart from its precise value).

One can equally well extract cutoff-insensitive information out of it. One possible way is to consider a variation of $\langle\rho_{-\mathbf{p}}\rho_{\mathbf{p}}\rangle|_{\nu=0}$ for $B\neq 0$ and $B=0$. Experimentally, this means measuring the vacuum weight for $B\neq 0$ and $B=0$ separately.

For $B=0$, the spectral weight for the graphene vacuum $|0\rangle$ with $m\to 0$ is written as

$$\frac{1}{\Omega}\langle 0|\rho_{-\mathbf{p}}\rho_{\mathbf{p}}|0\rangle^{B=0}=\lambda_s\lambda_v\int\frac{d^2\mathbf{k}}{(2\pi)^2}\frac{1}{2}(1-\cos\theta_{\mathbf{k+p},\mathbf{k}}), \tag{3.9}$$

which represents a collection of virtual transitions[17] from a negative-energy electron state with momentum $\mathbf{k}$ to a positive-energy state with $\mathbf{k}+\mathbf{p}$ via the charge density ρ. Here, $\theta_{\mathbf{k+p},\mathbf{k}}$ denotes the angle between $\mathbf{k}$ and $\mathbf{k}+\mathbf{p}$; $\lambda_s=2$ and $\lambda_v=2$ denote the spin and valley degeneracies. This is again ultraviolet divergent. For regularization, we cut off the $\mathbf{k}$ integral at $|\mathbf{k}|=\Lambda$ and choose the "Fermi momentum" Λ so that the Dirac sea accommodates the same number of electrons as in the $B\neq 0$ case, $N_{\rm sea}=\lambda_s\lambda_v\Lambda^2/4\pi\approx\lambda_s\lambda_v(N+1/2)/2\pi\ell^2$, i.e., $\Lambda^2\approx 2N/\ell^2$.

A direct calculation yields

$$\frac{1}{\Omega}\langle\rho_{-\mathbf{p}}\rho_{\mathbf{p}}\rangle^{B=0}=\lambda_s\lambda_v\frac{\mathbf{p}^2}{32\pi}\log\frac{16\Lambda^2}{\mathbf{p}^2}. \tag{3.10}$$

The cutoff (N) dependence, thus, correctly disappears from the difference $\delta\langle\rho\rho\rangle\equiv\langle\rho\rho\rangle^{B\neq 0}-\langle\rho\rho\rangle^{B=0}$, with the result

$$\delta\langle\rho_{-\mathbf{p}}\rho_{\mathbf{p}}\rangle|_{\nu=0}/\Omega=(\lambda_s/2\pi\ell^2)\,\delta I(z),$$

$$\delta I(z)=2e^{-z}I^{\psi}_{0\_}(z)-(z/4)\log(16N/z). \tag{3.11}$$

As seen from Fig. 1(b), the vacuum weight $\langle\rho_{-\mathbf{p}}\rho_{\mathbf{p}}\rangle|_{\nu=0}$ differs only slightly for $B\neq 0$ and $B=0$.

Actually, for all ν, the spectral weight $\langle\rho_{-\mathbf{p}}\rho_{\mathbf{p}}\rangle$ contains the vacuum fluctuations and the static structure factor $s(\mathbf{p})$ is necessarily cutoff dependent. A possible cutoff-independent measure in experiment is to compare the spectral weights for $\nu\neq 0$ and $\nu=0$ (with $B\neq 0$). Correspondingly, let us define by

$$\Delta s(\mathbf{p})=\{\langle\rho_{-\mathbf{p}}\rho_{\mathbf{p}}\rangle-\langle\rho_{-\mathbf{p}}\rho_{\mathbf{p}}\rangle|_{\nu=0}\}/N_e \tag{3.12}$$

the structure factor with the vacuum contribution subtracted. For the $\nu=2(2j+1)$ states (with $j=0,1,\ldots$), $\Delta s(\mathbf{p})=e^{-z}(\Delta I^{\psi}_j+\Delta I^{\chi}_j)/(2j+1)$ in terms of the cutoff-independent deviations $\Delta I^{\psi}_j=I^{\psi}_j-I^{\psi}_{0\_}$ and $\Delta I^{\chi}_j=I^{\chi}_j-I^{\chi}_{-1}$. For $j\geqslant 0$, they are rewritten as

$$\Delta I^{\chi}_j=\sum_{k=j+1}^{N}\sum_{n=0_+}^{j}\Gamma^{\chi}_{k,n}-\sum_{k=0_+}^{j}\sum_{n=1}^{N}\Gamma^{\chi}_{k,-n} \tag{3.13a}$$

$$\overset{m\to 0}{=}\sum_{n=1}^{j}F_n(z)e^{z}-\sum_{k=1}^{j}\sum_{n=0_+}^{j}\Gamma^{\chi}_{k,n}(z); \tag{3.13b}$$

analogously for ΔI^{ψ}_j. In reaching the second line, we have used the formulas (valid for $m\to 0$ and $N\to\infty$)

$$\sum_{k=1}^{N}\Gamma_{\pm k,0}(z)=\frac{1}{2}(e^{z}-1),$$

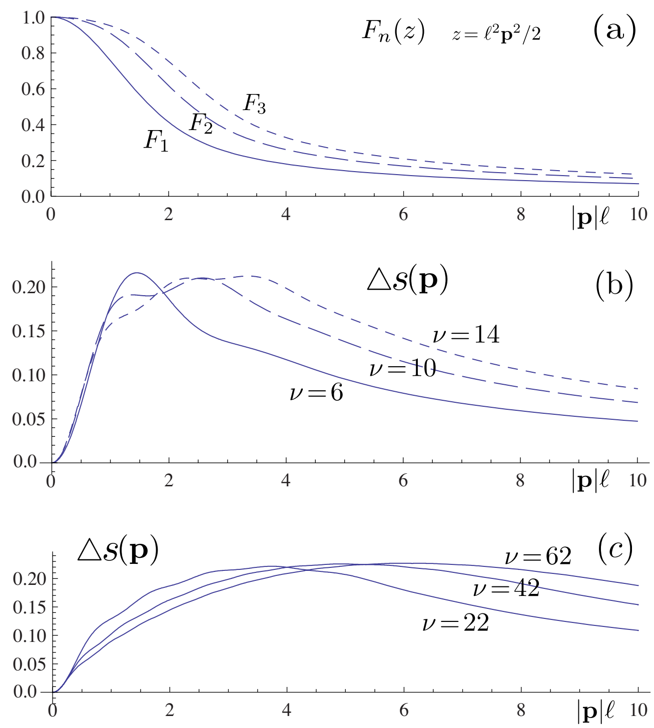

FIG. 2. (Color online) (a) $F_n(z)$. (b) Subtracted static structure factor $\Delta s(\mathbf{p})$ for $\nu=6$, 10, and 14. (c) $\Delta s(\mathbf{p})$ for $\nu=22$, 42, and 62.

$$\sum_{k=1}^{N}\Gamma_{k,n}(z)\overset{n\neq 0}{=}\frac{1}{2}[1+s_nF_{|n|}(z)]e^{z}-\frac{1}{4}\frac{z^{|n|}}{|n|!}, \tag{3.14}$$

where $z=\ell^2\mathbf{p}^2/2$ and

$$F_n(z)=e^{-z}\sum_{k=1}^{N}\mathrm{Re}[f_{n-1,k-1}(-\mathbf{p})f_{k,n}(\mathbf{p})]; \tag{3.15}$$

$F_n(0)=1$ and $F_n(z)\to 0$ as $z\to\infty$; the decrease is slower for larger n, as depicted in Fig. 2(a).

The subtracted structure factor is now written as

$$\Delta s(\mathbf{p})=\frac{2}{2j+1}\left[\sum_{n=1}^{j}F_n(z)-e^{-z}\sum_{k=1}^{j}\sum_{n=0_+}^{j}\Gamma^{\chi}_{k,n}(z)\right] \tag{3.16}$$

for $\nu=2(2j+1)$. In particular, $\Delta s(\mathbf{p})|_{\nu=2}=0$ and

$$\Delta s(\mathbf{p})|_{\nu=6}=\frac{2}{3}\left[F_1(z)-e^{-z}\left(1-\frac{1}{2}z+\frac{1}{4}z^2\right)\right]. \tag{3.17}$$

In Figs. 2(b) and 2(c), we plot $\Delta s(\mathbf{p})$ for $\nu=6$, 10, and 14 and for higher filling $\nu=22$, 42, and 62. There are some notable features: (1) $\Delta s(\mathbf{p})\to 0$ for $|\mathbf{p}|\to 0$. This is a consequence of the uniformity of charge at long wavelengths. (2) $\Delta s(\mathbf{p})\to 0$ for $|\mathbf{p}|\to\infty$. This shows that the large $|\mathbf{p}|$ portion of $s(\mathbf{p})$ is common to all ν and is governed by the vacuum weight $\langle\rho_{-p}\rho_{\mathbf{p}}\rangle|_{\nu=0}$. As a result, $s(\mathbf{p})$ itself would rise with $|\mathbf{p}|$, in sharp contrast to the standard behavior $s(\mathbf{p})\to 1$ as $|\mathbf{p}|\to\infty$ for nonrelativistic Hall electrons in Eq. (2.13). (3) $\Delta s(\mathbf{p})$ tends to saturate around 0.25 over a certain broad

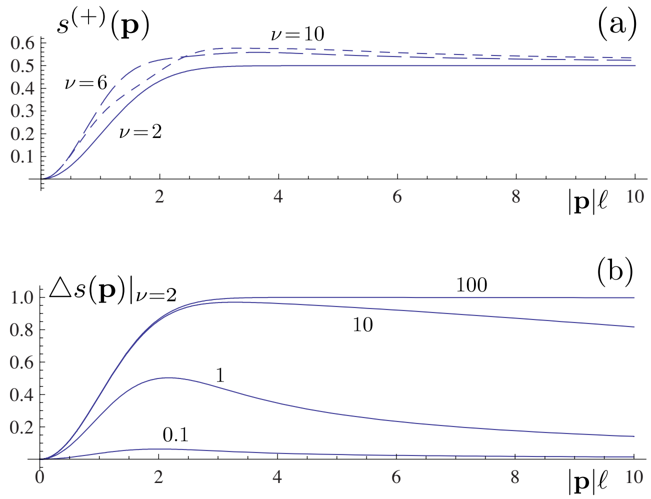

FIG. 3. (Color online) (a) Structure factor $s^{(+)}$ ($\mathbf{p}$ restricted to the positive-energy sector) for $\nu=2-10$. (b) $\Delta s(\mathbf{p})$ at $\nu=2$ develops as the mass gap becomes large $v_F m/\omega_c=0.1$, 1, 10, and 100.

range of $|\mathbf{p}|$ for higher filling factors ν. The subtracted structure factor $\Delta s(\mathbf{p})$, unlike $s(\mathbf{p})$, refers to quantum fluctuations of a finite number of filled levels above the vacuum and is governed by virtual transitions, newly allowed or suppressed by the presence of such levels, as seen from Eq. (3.13a). Saturation is a consequence of a competition between the transitions among positive-energy states and the suppressed vacuum polarization effect.

In this connection, let us try to retain only the positive-energy states and transitions among them. In Fig. 3(a), we plot a projected structure factor $s^{(+)}(\mathbf{p})$ calculated from the first term on the right-hand side of Eq. (3.13a). Note that

$$s^{(+)}(\mathbf{p}) \to 1/2 \quad \text{for } |\mathbf{p}| \to \infty, \tag{3.18}$$

i.e., even the positive-energy sector alone does not recover the $s(\mathbf{p})\to 1$ behavior of conventional QH systems. While $s^{(+)}(\mathbf{p})$ itself is not directly observable in experiment, this feature (3.18) would indicate indirectly that $\Delta s(\mathbf{p})$ is significantly smaller than $1/2$.

Some remarks on the influence of a mass gap are in order here. The relativistic treatment makes sense when the mass gap is tiny, $m\ell \ll 1$. In the nonrelativistic limit $m\ell \gg \sqrt{N}$ where the mass gap is large compared with the depth of the Dirac sea, in contrast, the Landau-level sum is limited to a finite interval $N \ll m^2\ell^2$ and ceases to yield a divergence. The virtual transition rates across the mass gap scale like $\Gamma_{k,-n} \propto 1/(m\ell)^2$ [while $\Gamma_{kn}\sim O(1)$ for $kn>0$], and the vacuum spectral weight $\langle \rho_{-\mathbf{p}}\rho_{\mathbf{p}}\rangle|_{\nu=0} \propto \Sigma_{k,n}\Gamma_{k,-n}$ tends to zero like $(m\ell)^{-2}\log(m^2\ell^2)\to 0$ as $m\ell\to\infty$. Figure 3(b) shows the manner how $\Delta s(\mathbf{p})|_{\nu=2}$ develops as the mass gap gets large for $v_F m/\omega_c=0.1\to 100$. The $\nu=2$ state is now distinguishable in $s(\mathbf{p})$ from the vacuum. In the $m\ell\to\infty$ limit, $\Delta s(\mathbf{p}) \approx s(\mathbf{p}) \approx s^{(+)}(\mathbf{p})$ recovers the standard behavior $s(\mathbf{p})\to 1$ as $|\mathbf{p}|\to\infty$ for all integer ν.

To explore the origin of the unconventional behavior (3.18), let us note the formula

$$\sum_{k=-\infty}^{\infty} |g_{kn}(\mathbf{p})|^2 = e^{\ell^2\mathbf{p}^2/2}, \tag{3.19}$$

which follows from Eq. (3.14). Actually, this formula is valid for $m\neq 0$ as well, since it is independently derived from the evaluation of the static weight $\langle G|\rho_{-\mathbf{p}}\rho_{\mathbf{p}}|G\rangle$ for a hypothetical state $|G\rangle$ with only one filled Landau level $\{n\}$ and all other levels empty. It is, thus, a consequence of completeness of physical states.

In the nonrelativistic limit, the Dirac sea effectively disappears. Then Eq. (3.19) is reduced to Eq. (2.12), which in turn leads to the behavior $s(\mathbf{p})\to 1$ for $|\mathbf{p}|\to\infty$.

For $m\to 0$, the large-$|\mathbf{p}|$ behavior of Eq. (3.19) is governed by the sums over $|k|\gg|n|$ so that

$$e^{-\ell^2\mathbf{p}^2/2}\sum_{k\sim 0}^{\infty} |g_{\pm k,n}(\mathbf{p})|^2 \to 1/2, \tag{3.20}$$

as seen also from Eq. (3.14). This precisely accounts for the behavior of $s^{(+)}(\mathbf{p})$ in Eq. (3.18). It is now clear that both the masslessness (or a tiny mass gap) of the quasiparticles and the presence of the Dirac sea are crucial for the unusual density correlations $s^{(+)}(\mathbf{p})\to 1/2$ at short distances. In other words, correlations fail to vanish at short distances because of particle-hole pair creation; the inability of localizing massless particles is a purely relativistic effect, underlying also the Klein paradox.[15]

Equation (3.19) may suggest that the vacuum spectral weight $\langle \rho_{-\mathbf{p}}\rho_{\mathbf{p}}\rangle|_{\nu=0}$ would have a contribution of $O(N)$ from the Dirac sea. It actually rises like $\ln N$, as we have seen in Eq. (3.7). Had we defined the "vacuum" static structure factor $s(\mathbf{p})$ with normalization by the total number of electrons in the Dirac sea, it would vanish like $s(\mathbf{p})\propto(\ln N)/N\to 0$. This shows again that, for massless particles and in the presence of the Dirac sea, the standard (classical) picture of particle correlations does not necessarily make sense.

Finally, we wish to comment on some peculiarities of the zero-energy $n=0_{\pm}$ levels, especially on how the Coulomb interaction $v_{\mathbf{p}}=2\pi\alpha/(\epsilon_b|\mathbf{p}|)$ affects their response. In the random-phase approximation (RPA), the effect of H^{Coul} of Eq. (2.2) is readily included[22] into the polarization function ($\sim -i\langle\rho\rho\rangle$),

$$P_{\mathrm{RPA}}(\mathbf{p},\omega) = P(\mathbf{p},\omega)/\{1 - v_{\mathbf{p}}P(\mathbf{p},\omega)\}, \tag{3.21}$$

once one knows the polarization function $P(\mathbf{p},\omega)$ for noninteracting Hall electrons. Suppose now that we start with the vacuum state and fill up the $n=0_+$ level to reach the $\nu=2$ state. Then $P(\mathbf{p},\omega)$ will change by an amount proportional to

$$\Delta P(\mathbf{p},\omega) \propto \sum_{k\geqslant 1}\frac{1}{\epsilon_k-\epsilon_{0_+}\pm\omega}\Gamma^{\chi}_{k,0_+}(\mathbf{p}) - \sum_{k\geqslant 1}\frac{1}{\epsilon_{0_+}-\epsilon_{-k}\pm\omega}\Gamma^{\chi}_{0_+,-k}(\mathbf{p}), \tag{3.22}$$

which is easily seen to vanish for zero mass gap $m\to 0$. This shows that the $\nu=0$ and $\nu=\pm 2$ states remain indistinguishable in density response even at the RPA level, indicating the robustness of the zero modes against perturbations. In particular, the static structure factor $s(\mathbf{p}) \sim \int d\omega P_{\text{RPA}}(\mathbf{p},\omega)$ and the dielectric function $\epsilon(\mathbf{p},\omega) = 1 - v_{\mathbf{p}} P(\mathbf{p},\omega)$ remain the same for $\nu=0$ and $\nu=\pm 2$.

IV. SUMMARY AND DISCUSSION

In this paper, we have studied the static structure factor for graphene in a strong magnetic field, with emphasis on revealing possible quantum signatures that distinguish graphene from conventional QH systems. In particular, for graphene, the vacuum state is a dielectric medium full of virtual particle-hole pairs. In other words, $\rho|\nu=0\rangle \neq 0$ for graphene, while $\rho|\nu=0\rangle = 0$ for standard QH systems. Correspondingly, the graphene vacuum state has a nonzero density spectral weight $\langle \rho_{-\mathbf{p}} \rho_{\mathbf{p}} \rangle|_{\nu=0}$, which actually diverges with the (cutoff) depth of the Dirac sea and generally rises with $|\mathbf{p}|$ significantly. Experimentally, the nonzero vacuum weight $\langle \rho_{-\mathbf{p}} \rho_{\mathbf{p}} \rangle|_{\nu=0}$ itself as well as its rise with $|\mathbf{p}|$, measured via inelastic light scattering, would be a clear signal of the quantum nature of the graphene vacuum state.

For graphene, the static structure factor $s(\mathbf{p})$ grows with $|\mathbf{p}|$ for all ν, in sharp contrast to the behavior $s(\mathbf{p}) \to 1$ as $|\mathbf{p}| \to \infty$ of conventional QH systems; this is because of pair creation at short distances. Actually, even the transitions among positive-energy states fail to recover the standard behavior and lead to $s^{(+)}(\mathbf{p}) \to 1/2$ for $|\mathbf{p}| \to \infty$, as noted in Eq. (3.18). This feature of $s^{(+)}(\mathbf{p})$ itself, unfortunately, is not directly observable. Nevertheless, for large filling factor ν, it competes with the vacuum polarization effect and would make the observable subtracted structure factor $\Delta s(\mathbf{p})$ (which is sensitive to quantum fluctuations of cutoff-independent longer wavelengths) saturate around $1/4$ over a certain broad range of $|\mathbf{p}|$; this could be a possible signature of the underlying dynamics. Such unusual features of particle correlations at short distances are a relativistic effect coming from massless Dirac particles in graphene.

Of special interest, in addition, are some peculiar features of the lowest ($n=0_{\pm}$) Landau levels of zero energy. Remarkably, they hardly affect the density response. The virtual transitions, both *allowed* and *suppressed* anew by the presence (or absence) of the $n=0_{\pm}$ levels, combine to leave the density response unchanged, and this compensation persists even when the Coulomb interaction is taken into account in the RPA. The graphene vacuum ($\nu=0$) state and the $\nu=\pm 2$ states would, thus, be indistinguishable in spectral weight $\langle \rho_{-\mathbf{p}} \rho_{\mathbf{p}} \rangle \sim s(\mathbf{p})$ as well as in electric susceptibility, although they are distinct in Hall conductance. Experimentally, one would expect a definite nonzero signal for $\langle \rho_{-\mathbf{p}} \rho_{\mathbf{p}} \rangle$ which largely remains the same over the range $|\nu| \leq 2$, except for some values of ν where nontrivial dynamics such as the fractional quantum Hall effect may come into play.

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